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New Differential Formulae Related to Hermite Polynomials and their Applications in Quantum Optics

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Abstract

In this work, based on quantum operator Hermite polynomials and Weyl's mapping rule, we find a generation function of the two-variable Hermite polynomials. And then, noting that the Weyl ordering is invariant under the similar transformations, we obtain another generalized differential expression related to the Hermite polynomials. Those identites can be applied to investigate the nonclasscial properties of quantum optical fields.

Keywords: Weyl mapping rule; Hermite polynomials; similar transformation
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1 Introduction

As the "language" of quantum mechanics, Dirac's bra-kets have come to represent a quantum world of abstract ideas and universal concepts and can get a far better understanding of quantum mechanics. Because conceptions in quantum mechanics quite differ from those in classical mechanics, it is inevitable that quantum mechanics must have its own mathematical symbols which are endowed with special physical meaning. For instance, in math [1], an inhomogeneous Fredholm equation (FE) of the first kind is written as $g(t) = \int_{-a}^{b} k(t,s) f(s) ds$, $K(t,s)$ is the continuous kernel function. In quantum mechanics [2, 3], we introduced an operator Fredholm equation defined as $G(a,a^\dagger) = \int_{-a}^{b} K(a,a^\dagger,q) F(q) dq$, in which the kernel function $K(a,a^\dagger,q)$ is a quantum operator, q is a real variable, a and $a^\dagger$ denote the annihilation and creation operator of a quantized radiation field. As is well known, integrations over the operators of type $|\rangle\langle|$ cannot be directly performed by Newton–Leibniz integration rule.

In Ref. [4], Fan proposed the technique of integration within an ordered product (IWOP) of operators which enables Newton–Leibniz integration rules directly working for Dirac's ket–bra operators with continuum variables. The technique of IWOP shows that the operator Fredholm equation (OFE) can directly perform integration if $K(a,a^\dagger,q)$ is an ordered product operator. An example of taking $K(a,a^\dagger,q) =: \exp(q-Q):$, we have

$$\int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi}} : \exp\left[-(q-Q)^2\right] : f(q) =: G(Q):, \qquad Q = \frac{a+a^\dagger}{\sqrt{2}}, \tag{1}$$

where $:\exp\left[-(q-Q)^2\right]:$ is the integral kernel, $a^\dagger$ commutes with a within the normally ordered symbol ": :". Noting that $\frac{1}{\sqrt{\pi}}:\exp\left[-(q-Q)^2\right]: = |q\rangle\langle q|$ and the completeness relation $\int_{-\infty}^{\infty} dq\,|q\rangle\langle q| = 1$, we can see

$$\int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi}} : \exp\left[-(q-Q)^2\right] : f(q) = \int_{-\infty}^{\infty} dq\,|q\rangle\langle q| f(q) = \int_{-\infty}^{\infty} dq\,|q\rangle\langle q| f(Q) = f(Q), \tag{2}$$

where $|q\rangle = \pi^{-1/4}\exp\left(-q^2/2 + \sqrt{2}qa^\dagger - a^{\dagger 2}/2\right)|0\rangle$ denotes the coordinate representation with its eigenfunction $Q|q\rangle = q|q\rangle$, and $:G(Q):$ is the normally ordered expansion of $f(Q)$. When $f(q) = H_n(q)$, we

have
$$\int_{-\infty}^{\infty}\frac{dq}{\sqrt{\pi}}:\ \exp\left[-\left(q-Q\right)^{2}\right]\ :H_{n}\left(q\right)=2^{n}:Q^{n}:\ , \tag{3}$$
where $2^{n}:Q^{n}:$ is the normally ordered form of the operator Hermite polynomials $H_{n}\left(Q\right)$. $H_{n}\left(q\right)$ is the single-variable Hermite polynomials with its generation function
$$\sum_{n=0}^{\infty}\frac{q^{n}}{n!}H_{n}\left(t\right)=\exp\left(2tq-q^{2}\right) \tag{4}$$
and $H_{n}\left(q\right)$ spans an orthonormal and complete function space, namely $\int_{-\infty}^{\infty}\frac{dq}{\sqrt{\pi}}e^{-q^{2}}H_{n}\left(q\right)H_{m}\left(q\right)=2^{n}n!\delta_{nm}$. Noting the annihilation operator $a=\left(Q+iP\right)/\sqrt{2}$, and $\left\langle q\right|a^{\dagger}=2^{-1/2}\left(q-\frac{d}{dq}\right)\left\langle q\right|$, we can derive the matrix element of $H_{n}\left(Q\right)$ in the coordinate representation and the vacuum state
$$\left\langle q\right|H_{n}\left(Q\right)\left|0\right\rangle=2^{n}\left\langle q\right|:Q^{n}:\ \left|0\right\rangle,$$
and then it follows
$$\begin{aligned}H_{n}\left(q\right)&=2^{n/2}e^{q^{2}/2}\left\langle q\right|:\ \left(a^{\dagger}+a\right)^{n}:\ \left|0\right\rangle=2^{n/2}e^{q^{2}/2}\left\langle q\right|a^{\dagger n}\left|0\right\rangle=e^{q^{2}/2}\left(q-\frac{d}{dq}\right)^{n}e^{-q^{2}/2}\\&=\left(-1\right)^{n}e^{q^{2}}\frac{d^{n}}{dq^{n}}e^{-q^{2}},\end{aligned} \tag{5}$$
which is just the differential expression of single-variable Hermite polynomials. The above derivation shows that Dirac's symbol and its own arithmetic rule can promote the development of the basic quantum theory.

In Ref.[5], author introduced a two-variable Hermite polynomials in complex space
$$H_{m,n}\left(\alpha,\alpha^{*}\right)=\sum_{l=0}^{\min(m,n)}\frac{m!n!}{l!\left(m-l\right)!\left(n-l\right)!}\left(-1\right)^{l}\alpha^{m-l}\alpha^{*n-l},\quad\alpha=q+ip, \tag{6}$$
whose generating function is
$$\sum_{m,n=0}^{\infty}\frac{t^{m}\tau^{n}}{m!n!}H_{m,n}\left(\alpha,\alpha^{*}\right)=\exp\left(-t\tau+t\alpha+\tau\alpha^{*}\right). \tag{7}$$
Two-variable Hermite polynomials can be applied in many fields of physics. For instance, $H_{m,n}\left(\alpha,\alpha^{*}\right)$ is proved to be the eigenmode of the complex fractional Fourier transform [6, 7, 8], so it may be observed in the light propagation in graded index (GRIN) medium, and this eigenmode is also the mechanism for two-dimensional Talbot effect demonstrated in GRIN medium [9].

Due to there exists some similarities between the generation function of single-variable Hermite polynomials $H_{n}\left(x\right)$ and that of two-variable complex Hermite polynomials $H_{m,n}\left(\alpha,\alpha^{*}\right)$, it is interesting to see that the differential expression of $H_{m,n}\left(\alpha,\alpha^{*}\right)$ is similar to that of $H_{n}\left(x\right)$. In this work, by virtue of Weyl's mapping rule and quantum operator Hermite polynomials, we obtain the differential expression of $H_{m,n}\left(\alpha,\alpha^{*}\right)$ and another generalized forms. So our work is arranged as follows. In Sec. 2, we briefly introduce Weyl correspondence rule, which is related to the Weyl ordering and the technique of integration within Weyl ordered product (IWWP) of operators. We reveal that quantum correspondence operator of classical function $f\left(p,q\right)$ can directly be obtianed by replacing q and p in $f\left(p,q\right)$ by Q and P with the function form invariant. In Sec. 3 we introduce the two-variable Hermite function. By formulizing the Weyl correspondence, we derive a differential expression of $H_{m,n}\left(\alpha,\alpha^{*}\right)$. Noting that the Weyl ordering is invariant under the similar transformations, the generalized differential expressions are obtained in Sec. 4. Enlightened by those new identities, in Sec. V we investigate the nonclassical properties of an excited squeezed vacuum state, which can be generated in quantum systems with restricted dimensions.

2 Weyl Ordering and Weyl Correspondence Rule

As is well known, the Weyl correspondence rule [10, 11], i.e.
$$F\left(P,Q\right)=\int\int dqdpf\left(p,q\right)\Delta\left(q,p\right), \tag{8}$$

is a recipe for quantizing a classical function $f(p,q)$ defined in classical phase space as a quantum correspond operator $F(P,Q)$. $\Delta(q,p)$ is the Wigner operator difined as

$$\Delta(p,q)=\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{ipu}\left|q+\frac{u}{2}\right\rangle\left\langle q-\frac{u}{2}\right|. \tag{9}$$

When $f(p,q)=q^m p^r$, from Eq. (8) its quantum corresponding operators is

$$q^m p^r\rightarrow\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}Q^{m-l}P^rQ^l=\left(\frac{1}{2}\right)^m\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}\vdots\, Q^{m-l}P^rQ^l\,\vdots=\vdots\, Q^mP^r\,\vdots, \tag{10}$$

in which $\vdots\;\vdots$ denotes the Weyl ordering symbol, and Eq. (10) tell us that the Weyl ordered correspond operator of classcial function $f(p,q)$ is obtained by just replacing p,q in $f(p,q)$ by P,Q with the function form invariant. As one of the definite operator orderings (such as normal ordering, anti-normal ordering and Weyl ordering), Weyle ordering is a useful one and within Weyl ordering symbol $\vdots\;\vdots$ the Bose operators are permutable. Enlighted by the technique of IWOP, [13] proposed the technique of integration within Weyl ordered product (IWWP) of operators. From (9) we obtain the Weyl ordering form of the Wigner opertor

$$\Delta(p,q)=\vdots\,\delta(p-P)\,\delta(q-Q)\,\vdots. \tag{11}$$

Therefore one can easily obtain quantum correspond operator of classical function $h(p,q)$ by replacing $q\rightarrow Q$, $p\rightarrow P$, i.e.

$$F(P,Q)=\vdots\,h(P,Q)\,\vdots=\int\int_{-\infty}^{\infty}dpdqh(p,q)\,\Delta(p,q). \tag{12}$$

Eq. (1) can tell us that the Weyl correspondence rule is also an OFE. Noting that

$$\begin{aligned}\mathrm{Tr}\left[\Delta(p_1,q_1)\,\Delta(p_2,q_2)\right] &= \int\frac{d^2z}{\pi}\langle z|\left[\Delta(p_1,q_1)\,\Delta(p_2,q_2)\right]|z\rangle\\ &= \frac{1}{2\pi}\delta(q_1-q_2)\,\delta(p_1-p_2),\end{aligned} \tag{13}$$

it then follows that the reciprocal relation of the Weyl correspondence rule is

$$2\pi Tr\left[F(P,Q)\,\Delta(q,p)\right]=2\pi\mathrm{Tr}\left[\int\int dq_1dp_1h(p_1,q_1)\,\Delta(p_1,q_1)\,\Delta(p,q)\right]=h(p,q). \tag{14}$$

In many cases, taking $\alpha=(q+ip)/\sqrt{2}$, the Wigner operator in (9) can be rewritten as

$$\begin{aligned}\Delta(p,q) &\rightarrow \Delta(\alpha,\alpha^*)=\int\frac{d^2z}{\pi^2}|\alpha+z\rangle\langle\alpha-z|\,e^{\alpha z^*-\alpha^*z}\\ &= \frac{1}{\pi}{:}\,\exp\left[-2\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]{:}=\frac{1}{2}\vdots\,\delta\left(a^\dagger-\alpha^*\right)\delta(a-\alpha)\,\vdots.\end{aligned} \tag{15}$$

It then follows that the Weyl correspondence formula in Eq. (12) can be recast to

$$G\left(a,a^\dagger\right)=\vdots\,f\left(a,a^\dagger\right)\,\vdots=2\int\int dqdpf(\alpha,\alpha^*)\,\Delta(\alpha,\alpha^*), \tag{16}$$

with its reciprocal relation

$$2\pi Tr\left[G\left(a,a^\dagger\right)\Delta(\alpha,\alpha^*)\right]=2\pi Tr\left[\vdots\,f\left(a,a^\dagger\right)\,\vdots\,\Delta(\alpha,\alpha^*)\right]=f(\alpha,\alpha^*). \tag{17}$$

Especially, when $G\left(a,a^\dagger\right)=\rho\left(a,a^\dagger\right)$ describes a density of states for an interesting quantum system, from the above reciprocal relation we can see

$$2\pi Tr\left[\rho\left(a,a^\dagger\right)\Delta(\alpha,\alpha^*)\right]=W(\alpha,\alpha^*), \tag{18}$$

$W(\alpha,\alpha^*)$ denotes a quasi-probability distribution Wigner function. Thus Eq. (18) can also be called the Wigner-Weyl correspondence rule.

3 New Differential Expression of Two-variable Hermite Polynomials $H_{m,n}(\alpha,\alpha^*)$

In quantum theory, $H_{m,n}(\alpha,\alpha^*)$ is the generalized Bargmann representation of the two-mode Fock state in the bipartite entangled state representation [14], i.e.

$$|m,n\rangle = \frac{a^{\dagger m}b^{\dagger m}}{\sqrt{m!n!}}|0,0\rangle \to \frac{1}{\sqrt{m!n!}}H_{m,n}(\xi,\xi^*)e^{-\frac{|\xi|^2}{2}},$$

and it spans an orthonormal and complete function space,

$$2\int\int\frac{d^2\xi}{\pi}e^{-2|\xi|^2}H_{m,n}\left(\sqrt{2}\xi,\sqrt{2}\xi^*\right)H^*_{m',n'}\left(\sqrt{2}\xi,\sqrt{2}\xi^*\right) = \sqrt{m!n!m'!n'!}\delta_{m,m'}\delta_{n,n'}. \tag{19}$$

Eq.(19) indicates any one function $f(\alpha,\alpha^*)$ can be expanded by those orthogonal basis,

$$f(\alpha,\alpha^*) = \sum_{m,n=0}^{\infty}C_{m,n}H^*_{m,n}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right). \tag{20}$$

where $C_{m,n}$ is a constant to be determined by follow derivation. From its generation function shown in (7), we can also expand the normally ordered form of Wigner operator in Eq. (15) as

$$\Delta(\alpha,\alpha^*) = \frac{1}{\pi}:\exp\left[-2\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]: = \frac{1}{\pi}e^{-2|\alpha|^2}:\sum_{m,n=0}^{\infty}\frac{\sqrt{2^{m+n}}a^{\dagger m}a^n}{m!n!}H_{m,n}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right):. \tag{21}$$

Substituting (20) and (21) into (16), we have

$$\begin{aligned} G\left(a^\dagger,a\right) &= 2\int\frac{d^2\alpha}{\pi}e^{-2|\alpha|^2}:\sum_{m,n=0}^{\infty}\frac{\sqrt{2^{m+n}}a^{\dagger m}a^n}{m!n!}H_{m,n}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right):\sum_{m',n'=0}^{\infty}C_{m',n'}H^*_{m',n'}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right) \\ &= 2:\sum_{m,n=0}^{\infty}\sum_{m',n'=0}^{\infty}C_{m',n'}\frac{\sqrt{2^{m+n}}a^{\dagger m}a^n}{m!n!}:\int\frac{d^2\alpha}{\pi}e^{-2|\alpha|^2}H_{m,n}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right)H^*_{m',n'}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right) \\ &= :\sum_{m,n=0}^{\infty}C_{m,n}\sqrt{2^{m+n}}a^{\dagger m}a^n: \equiv :F\left(a^\dagger,a\right):. \end{aligned} \tag{22}$$

Taking the coherent state expectation values of (22), we have

$$\langle\alpha|:F\left(a^\dagger,a\right):|\alpha\rangle = F(\alpha^*,\alpha) = \langle\alpha|:\sum_{m,n=0}^{\infty}C_{m,n}\sqrt{2^{m+n}}a^{\dagger m}a^n:|\alpha\rangle = \sum_{m,n=0}^{\infty}C_{m,n}\sqrt{2^{m+n}}(\alpha^*)^m\alpha^n,$$

and then

$$C_{m,n} = \frac{1}{m!n!\sqrt{2^{m+n}}}\frac{\partial^m}{\partial\alpha^{*m}}\frac{\partial^n}{\partial\alpha^n}F(\alpha^*,\alpha)\bigg|_{\alpha=0}.$$

Therefore, from (20) we obtain

$$F(\alpha,\alpha^*) = \sum_{m,n=0}^{\infty}H_{m,n}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right)\frac{1}{m!n!\sqrt{2^{m+n}}}\frac{\partial^m}{\partial\alpha^{*m}}\frac{\partial^n}{\partial\alpha^n}F(\alpha^*,\alpha)\bigg|_{\alpha=0}. \tag{23}$$

This is a new formula for deriving Weyl's classical correspondence of normally ordered quantum operators. For example, when $G_1\left(a^\dagger,a\right) =: F_1\left(a^\dagger,a\right): =: a^{\dagger m}a^n:$, from Eq. (23) we have

$$g_1(\alpha,\alpha^*) = \frac{1}{\sqrt{2^{m+n}}}H_{m,n}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right). \tag{24}$$

And then considering the reciprocal relation in (17), we can see

$$g_1(\alpha,\alpha^*) = 2\pi\mathrm{Tr}\left[G_1\left(a^\dagger,a\right)\Delta(\alpha,\alpha^*)\right] = 2\pi\mathrm{Tr}\left[a^{\dagger m}a^n\Delta(\alpha,\alpha^*)\right] = \frac{1}{\sqrt{2^{m+n}}}H_{m,n}\left(\sqrt{2}\alpha,\sqrt{2}\alpha^*\right)$$

On the other hand, given any operator $F\left(a,a^{\dagger}\right)$, its classical correspondence can also be obtained by utilizing the following formular (see Appendix for details)

$$f\left(\alpha^{*},\alpha\right)=e^{2|\alpha|^{2}}\int\frac{d^{2}\beta}{\pi}\left\langle-\beta\right|F\left(a^{\dagger},a\right)\left|\beta\right\rangle e^{2\left(\alpha\beta^{*}-\alpha^{*}\beta\right)}. \tag{25}$$

Substituting $G_{1}\left(a^{\dagger},a\right)=:a^{\dagger m}a^{n}:$ into (25), we have

$$\begin{aligned}
g_{1}\left(\alpha,\alpha^{*}\right) &= e^{2|\alpha|^{2}}\int\frac{d^{2}\beta}{\pi}\left\langle-\beta\right|G_{1}\left(a^{\dagger},a\right)\left|\beta\right\rangle e^{2\left(\alpha\beta^{*}-\alpha^{*}\beta\right)}=e^{2|\alpha|^{2}}\int\frac{d^{2}\beta}{\pi}\left\langle-\beta\right|:a^{\dagger m}a^{n}:\left|\beta\right\rangle e^{2\left(\alpha\beta^{*}-\alpha^{*}\beta\right)}\\
&= e^{2|\alpha|^{2}}\int\frac{d^{2}\beta}{\pi}\left(-\beta^{*}\right)^{m}\beta^{n}e^{-2|\beta|^{2}+2\left(\alpha\beta^{*}-\alpha^{*}\beta\right)}.
\end{aligned}$$

Making the substitution $\sqrt{2}\beta\rightarrow\beta,\sqrt{2}\beta^{*}\rightarrow\beta^{*}$, the above equation can be derived as

$$\begin{aligned}
g_{1}\left(\alpha,\alpha^{*}\right) &= \frac{\left(-1\right)^{m}e^{2|\alpha|^{2}}}{\left(\sqrt{2}\right)^{m+n}}\int\frac{d^{2}\beta}{\pi}\beta^{*m}\beta^{n}\exp\left[-\left|\beta\right|^{2}+\sqrt{2}\left(\alpha\beta^{*}-\alpha^{*}\beta\right)\right]\\
&= \frac{\left(-1\right)^{m-n}e^{2|\alpha|^{2}}}{2^{m+n}}\frac{\partial^{m}}{\partial\alpha^{m}}\frac{\partial^{n}}{\partial\alpha^{*n}}\int\frac{d^{2}\beta}{\pi}\exp\left[-\left|\beta\right|^{2}+\sqrt{2}\left(\alpha\beta^{*}-\alpha^{*}\beta\right)\right].
\end{aligned}$$

Utilizing the following integral formula

$$\int\frac{d^{2}\alpha}{\pi}\exp\left[h\left|\alpha\right|^{2}+s\alpha+\eta\alpha^{*}\right]=\frac{1}{h}\exp\left[-\frac{s\eta}{h}\right],\quad\mathrm{Re}\left[h\right]<0,$$

we can obtain

$$g_{1}\left(\alpha,\alpha^{*}\right)=\frac{\left(-1\right)^{m-n}e^{2|\alpha|^{2}}}{2^{m+n}}\frac{\partial^{m}}{\partial\alpha^{m}}\frac{\partial^{n}}{\partial\alpha^{*n}}\exp\left(-2\left|\alpha\right|^{2}\right). \tag{26}$$

Comparing Eq. (26) and (24), and setting $\sqrt{2}\alpha\rightarrow\alpha,\sqrt{2}\alpha^{*}\rightarrow\alpha^{*}$ we can derive

$$H_{m,n}\left(\alpha,\alpha^{*}\right)=\left(-1\right)^{m-n}e^{|\alpha|^{2}}\frac{\partial^{m}}{\partial\alpha^{*m}}\frac{\partial^{n}}{\partial\alpha^{n}}\exp\left(-\left|\alpha\right|^{2}\right), \tag{27}$$

which is just the differential form for the generation function of two-variable Hermite polynomials, and for the especial case of $m=n$

$$H_{m,m}\left(\alpha,\alpha^{*}\right)=\exp\left(\left|\alpha\right|^{2}\right)\frac{\partial^{2m}}{\partial\alpha^{*m}\alpha^{m}}\exp\left(-\left|\alpha\right|^{2}\right).$$

4 Generalized Differential Expressions related to Hermite Polynomials

In order to generate non-symmetric quantum mechanical representation, in Ref.[18] authors proposed a non-unitary operator $\hat{U}$, defined as

$$\begin{aligned}
\hat{U} &= \frac{1}{\sqrt{\mu}}\int\frac{d^{2}z}{\pi}\left|\begin{pmatrix}\mu&\nu\\ \sigma&\tau\end{pmatrix}\begin{pmatrix}z\\ z^{*}\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}z\\ z^{*}\end{pmatrix}\right|\\
&= \frac{1}{\sqrt{\mu}}:\exp\left[-\frac{\nu}{2\mu}a^{\dagger2}+\left(\frac{1}{\mu}-1\right)a^{\dagger}a+\frac{\sigma}{2\mu}a^{2}\right]:,
\end{aligned} \tag{28}$$

which is a quantum operator image of the classical symplectic transformation $\left(z,z^{*}\right)\rightarrow\left(\mu z+\nu z^{*},\sigma z+\tau z^{*}\right)$ in phase space, and where $\left|\begin{pmatrix}z\\ z^{*}\end{pmatrix}\right\rangle=\exp\left(za^{\dagger}-z^{*}a\right)\left|0\right\rangle$ denotes the coherent state [19] and $\left|\begin{pmatrix}\mu&\nu\\ \sigma&\tau\end{pmatrix}\begin{pmatrix}z\\ z^{*}\end{pmatrix}\right\rangle=\left|\begin{pmatrix}\mu z+\nu z^{*}\\ \sigma z+\tau z^{*}\end{pmatrix}\right\rangle=\exp\left[\left(\mu z+\nu z^{*}\right)a^{\dagger}-\left(\sigma z+\tau z^{*}\right)a\right]\left|0\right\rangle$. The invers of $\hat{U}$ reads as

$$\begin{aligned}
\hat{U}^{-1} &= \sqrt{\mu}\int\frac{d^{2}z}{\pi}\left|\begin{pmatrix}z\\ z^{*}\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\tau&-\nu\\ -\sigma&\mu\end{pmatrix}\begin{pmatrix}z\\ z^{*}\end{pmatrix}\right|\\
&= \frac{1}{\sqrt{\tau}}:\exp\left[\frac{\nu}{2\tau}a^{\dagger2}+\left(\frac{1}{\tau}-1\right)a^{\dagger}a-\frac{\sigma}{2\tau}a^{2}\right]:.
\end{aligned} \tag{29}$$

From (28) and (29), we can find $\hat{U}^{\dagger} \neq \hat{U}^{-1}$ and $\hat{U}$ engenders a similar transformation

$$\hat{U} a \hat{U}^{-1} = \mu a + \nu a^{\dagger},\ \hat{U} a^{\dagger} \hat{U}^{-1} = \sigma a + \tau a^{\dagger} \tag{30}$$

and its invers transformation

$$\hat{U}^{-1} a \hat{U} = \tau a - \nu a^{\dagger},\ \hat{U}^{-1} a^{\dagger} \hat{U} = \mu a^{\dagger} - \sigma a, \tag{31}$$

where four complex parameters satisfies $\mu\tau - \nu\sigma = 1$ for keeping $\left[\mu a + \nu a^{\dagger},\ \sigma a + \tau a^{\dagger}\right] = 1$.Also it is important to note that the Weyl ordering is invariant under the similar transformations [4, 15, 16], i.e.

$$\hat{U} G\left(a^{\dagger}, a\right) \hat{U}^{-1} = \hat{U} \ \vdots\ g\left(a^{\dagger}, a\right) \ \vdots\ \hat{U}^{-1} = \ \vdots\ g\left(\sigma a + \tau a^{\dagger}, \mu a + \nu a^{\dagger}\right) \ \vdots\ . \tag{32}$$

4.1 Generalized Differential Expression Related to the Product of Two Single-variable Hermite Polynomials

Supposing an operator $G_2\left(a^{\dagger}, a\right) = a^{\dagger m} \hat{U} |0\rangle \langle 0| \hat{U}^{-1} a^{n}$, and from Eqs. (17) and (31), we can express the classical Weyl correspondence of $G_2\left(a^{\dagger}, a\right)$ as

$$\begin{aligned} g_2\left(\alpha^*, \alpha\right) &= 2\pi \mathrm{Tr}\left[G_2\left(a^{\dagger}, a\right) \Delta\left(\alpha, \alpha^*\right)\right] = 2\pi \mathrm{Tr}\left[a^{\dagger m} \hat{U} |0\rangle \langle 0| \hat{U}^{-1} a^{n} \Delta\left(\alpha, \alpha^*\right)\right] \\ &= 2\pi \mathrm{Tr}\left[\hat{U}\left(\mu a^{\dagger} - \sigma a\right)^{m} |0\rangle \langle 0| \left(\tau a - \nu a^{\dagger}\right)^{n} \hat{U}^{-1} \Delta\left(\alpha, \alpha^*\right)\right]. \end{aligned} \tag{33}$$

By virtue of the following formular

$$\left(f a + g a^{\dagger}\right)^{n} = \left(-i\sqrt{\frac{fg}{2}}\right)^{n} : H_n\left(i\sqrt{\frac{f}{2g}} a + i\sqrt{\frac{g}{2f}} a^{\dagger}\right) : ,$$

we can see

$$\left(\mu a^{\dagger} - \sigma a\right)^{m} = \left(\sqrt{\frac{\sigma\mu}{2}}\right)^{m} : H_m\left(-\sqrt{\frac{\sigma}{2\mu}} a - \sqrt{\frac{\mu}{2\sigma}} a^{\dagger}\right) : \tag{34}$$

$$\left(\tau a - \nu a^{\dagger}\right)^{n} = \left(\sqrt{\frac{\nu\tau}{2}}\right)^{n} : H_n\left(-\sqrt{\frac{\tau}{2\nu}} a - \sqrt{\frac{\nu}{2\tau}} a^{\dagger}\right) : . \tag{35}$$

Therefore we have

$$\begin{aligned} &\left(-\sigma a + \mu a^{\dagger}\right)^{m} |0\rangle \langle 0| \left(\tau a - \nu a^{\dagger}\right)^{n} \\ = &\left(\sqrt{\frac{\sigma\mu}{2}}\right)^{m} \left(\sqrt{\frac{\nu\tau}{2}}\right)^{n} : H_m\left(-\sqrt{\frac{\mu}{2\sigma}} a^{\dagger}\right) \exp\left[-a^{\dagger} a\right] H_n\left(-\sqrt{\frac{\tau}{2\nu}} a\right) : , \end{aligned} \tag{36}$$

where $|0\rangle \langle 0| =: \exp\left[-a^{\dagger} a\right] :$. Utilizing the following formular (for details in Appendix)

$$F\left(a^{\dagger}, a\right) = \ \vdots\ f\left(a^{\dagger}, a\right) \ \vdots\ = 2\int \frac{d^2 z}{\pi} \ \vdots\ \langle -z| F\left(a^{\dagger}, a\right) |z\rangle \exp\left[2\left(a z^* - a^{\dagger} z + a^{\dagger} a\right)\right] \ \vdots\ , \tag{37}$$

we can obtain the Weyl ordering form of $\left(-\sigma a + \mu a^{\dagger}\right)^{m} |0\rangle \langle 0| \left(\tau a - \nu a^{\dagger}\right)^{n}$, namely

$$\begin{aligned} &\left(-\sigma a + \mu a^{\dagger}\right)^{m} |0\rangle \langle 0| \left(\tau a - \nu a^{\dagger}\right)^{n} = 2\left(\sqrt{\frac{\sigma\mu}{2}}\right)^{m} \left(-\sqrt{\frac{\nu\tau}{2}}\right)^{n} \\ &\times \ \vdots\ \int \frac{d^2 z}{\pi} H_m\left(\sqrt{\frac{\mu}{2\sigma}} z^*\right) H_n\left(\sqrt{\frac{\tau}{2\nu}} z\right) \exp\left[-|z|^2 - 2a^{\dagger} z + 2a z^* + 2a^{\dagger} a\right] \ \vdots\ . \end{aligned} \tag{38}$$

An explicit integral formula proved in Ref. [17]

$$\int \frac{d^2 z}{\pi} H_m\left(z^*\right) H_n\left(z\right) \exp\left[-\left(z - \lambda\right)\left(z^* - \lambda^*\right)\right] = \sum_{l=0}^{\min[m,n]} \frac{2^{2l} m! n!}{l! \left(m-l\right)! \left(n-l\right)!} H_{m-l}\left(\lambda^*\right) H_{n-l}\left(\lambda\right) \tag{39}$$

can be utilized for the derivation of Eq.(38). Furtherly, we have

$$\int \frac{d^2 z}{\pi} H_m \left(\alpha z^*\right) H_n \left(\beta z\right) \exp\left[-\left(z-\lambda\right)\left(z^*-\lambda^*\right)\right] = \sum_{l=0}^{\min[m,n]} \frac{\left(4\alpha\beta\right)^l m!n!}{l!\left(m-l\right)!\left(n-l\right)!} H_{m-l}\left(\alpha\lambda^*\right) H_{n-l}\left(\beta\lambda\right), \quad (40)$$

then it follows

$$\begin{aligned} \left(-\sigma a+\mu a^\dagger\right)^m |0\rangle\langle 0| \left(\tau a-\nu a^\dagger\right)^n &= 2m!n! \left(-\sqrt{\frac{\sigma\mu}{2}}\right)^m \left(-\sqrt{\frac{\nu\tau}{2}}\right)^n \\ \times \vdots \sum_{l=0}^{\min[m,n]} \frac{\left(-2\sqrt{\frac{\mu\tau}{\sigma\nu}}\right)^l}{l!\left(m-l\right)!\left(n-l\right)!} & H_{m-l}\left(\sqrt{\frac{2\mu}{\sigma}} a^\dagger\right) H_{n-l}\left(\sqrt{\frac{2\tau}{\nu}} a\right) \exp\left[-2a^\dagger a\right] \vdots . \end{aligned} \quad (41)$$

Noting that Weyl ordering is invariant under the similar transformations shown in Eq. (32), and using the Weyl correspondence formula in Eq. (17), we can obtain the classical correspondence of $G_2\left(a^\dagger, a\right)$,

$$\begin{aligned} g_2\left(\alpha^*, \alpha\right) &= 2\pi \mathrm{Tr}\left[G_2\left(a^\dagger, a\right)\Delta\left(\alpha, \alpha^*\right)\right] = 2\pi \mathrm{Tr}\left[a^{\dagger m}\hat{U}|0\rangle\langle 0|\hat{U}^{-1}a^n\Delta\left(\alpha,\alpha^*\right)\right] \\ &= 2m!n! \left(-\sqrt{\frac{\sigma\mu}{2}}\right)^m \left(-\sqrt{\frac{\nu\tau}{2}}\right)^n \sum_{l=0}^{\min[m,n]} \frac{\left(-2\sqrt{\frac{\mu\tau}{\sigma\nu}}\right)^l}{l!\left(m-l\right)!\left(n-l\right)!} H_{m-l}\left[\sqrt{\frac{2\mu}{\sigma}}\left(\sigma\alpha+\tau\alpha^*\right)\right] \\ & \times H_{n-l}\left[\sqrt{\frac{2\tau}{\nu}}\left(\mu\alpha+\nu\alpha^*\right)\right] \exp\left[-2\left(\sigma\alpha+\tau\alpha^*\right)\left(\mu\alpha+\nu\alpha^*\right)\right]. \end{aligned} \quad (42)$$

From Eq. (25), the classical correspondence of $G_2\left(a^\dagger, a\right)$ can also be expressed as

$$\begin{aligned} g_2\left(\alpha^*, \alpha\right) &= e^{2|\alpha|^2} \int \frac{d^2\beta}{\pi} \langle -\beta| a^{\dagger m}\hat{U}|0\rangle\langle 0|\hat{U}^{-1}a^n|\beta\rangle e^{2\left(\alpha\beta^*-\alpha^*\beta\right)} \\ &= \frac{1}{\sqrt{\mu\tau}} e^{2|\alpha|^2} \int \frac{d^2\beta}{\pi} \left(-\beta^*\right)^m \beta^n \exp\left[-|\beta|^2 - 2\alpha^*\beta + 2\alpha\beta^* - \frac{\sigma}{2\tau}\beta^2 - \frac{\nu}{2\mu}\beta^{*2}\right] \\ &= \frac{e^{2|\alpha|^2}}{\sqrt{\mu\tau}} \frac{(-1)^{m+n}}{2^{m+n}} \frac{\partial^m}{\partial\alpha^m} \frac{\partial^n}{\partial\alpha^{*n}} \int \frac{d^2\beta}{\pi} \exp\left[-|\beta|^2 - 2\alpha^*\beta + 2\alpha\beta^* - \frac{\sigma}{2\tau}\beta^2 - \frac{\nu}{2\mu}\beta^{*2}\right]. \end{aligned} \quad (43)$$

And then using the following integral formula

$$\int \frac{d^2\alpha}{\pi} \exp\left[h|\alpha|^2 + s\alpha + \eta\alpha^* + f\alpha^2 + g\alpha^{*2}\right] = \frac{1}{\sqrt{h^2-4fg}} \exp\left[\frac{-hs\eta + s^2 g + \eta^2 f}{h^2-4fg}\right],$$

whose convergence conditions are $\mathrm{Re}(h \pm f \pm g) < 0$ and $\mathrm{Re}\left(\frac{h^2-4fg}{h\pm f\pm g}\right) < 0$, we have

$$g_2\left(\alpha^*, \alpha\right) = e^{2|\alpha|^2} \frac{(-1)^{m+n}}{2^{m+n}} \frac{\partial^m}{\partial\alpha^m} \frac{\partial^n}{\partial\alpha^{*n}} \exp\left[-4\mu\tau|\alpha|^2 - 2\nu\tau\alpha^{*2} - 2\sigma\mu\alpha^2\right]. \quad (44)$$

Comparing Eqs. (42) and (44), we can see

$$\begin{aligned} & \exp\left[4\mu\tau|\alpha|^2 + 2\nu\tau\alpha^{*2} + 2\sigma\mu\alpha^2\right] \frac{\partial^m}{\partial\alpha^m} \frac{\partial^n}{\partial\alpha^{*n}} \exp\left[-4\mu\tau|\alpha|^2 - 2\nu\tau\alpha^{*2} - 2\sigma\mu\alpha^2\right] \\ = \; & 2\left(2\mu\sigma\right)^{\frac{m}{2}} \left(2\tau\nu\right)^{\frac{n}{2}} \sum_{l=0}^{\min[m,n]} \binom{m}{l}\binom{n}{l} l! \left(-\sqrt{\frac{4\mu\tau}{\sigma\nu}}\right)^l \\ & \times H_{m-l}\left[\sqrt{\tfrac{2\mu}{\sigma}}\left(\sigma\alpha+\tau\alpha^*\right)\right] H_{n-l}\left[\sqrt{\tfrac{2\tau}{\nu}}\left(\mu\alpha+\nu\alpha^*\right)\right]. \end{aligned} \quad (45)$$

which is a generalized differential formula related to the product of two single-variable Hermite polynomials. If taking $\tau=\mu^*$, $\sigma=\nu^*$ to meet $|\mu|^2-|\nu|^2=1$, $\hat{U}$ is unitary. Thus Eq. (45) reduces to

$$\begin{aligned} & \exp\left[4|\mu|^2|\alpha|^2 + 2\mu^*\nu\alpha^{*2} + 2\mu\nu^*\alpha^2\right] \frac{\partial^m}{\partial\alpha^m} \frac{\partial^n}{\partial\alpha^{*n}} \exp\left[-4|\mu|^2|\alpha|^2 - 2\mu^*\nu\alpha^{*2} - 2\mu\nu^*\alpha^2\right] \\ = \; & 2\left(2\mu\nu^*\right)^{\frac{m}{2}} \left(2\mu^*\nu\right)^{\frac{n}{2}} \sum_{l=0}^{\min[m,n]} \binom{m}{l}\binom{n}{l} l! \left(-\frac{2|\mu|}{|\nu|}\right)^l \\ & \times H_{m-l}\left[\sqrt{\tfrac{2\mu}{\nu^*}}\left(\nu^*\alpha+\mu^*\alpha^*\right)\right] H_{n-l}\left[\sqrt{\tfrac{2\mu^*}{\nu}}\left(\mu\alpha+\nu\alpha^*\right)\right]. \end{aligned} \quad (46)$$

Especially, for $m=n$, Eq. (46) is

$$\begin{aligned}
&\exp\left[4\left|\mu\right|^{2}\left|\alpha\right|^{2}+2\mu^{*}\nu\alpha^{*2}+2\mu\nu^{*}\alpha^{2}\right]\frac{\partial^{m}}{\partial\alpha^{m}}\frac{\partial^{m}}{\partial\alpha^{*m}}\exp\left[-4\left|\mu\right|^{2}\left|\alpha\right|^{2}-2\mu^{*}\nu\alpha^{*2}-2\mu\nu^{*}\alpha^{2}\right] \\
=\;& 2^{m+1}\left|\mu\right|^{m}\left|\nu\right|^{m}\sum_{l=0}^{m}\binom{m}{l}\binom{m}{l}l!\left(-\frac{2\left|\mu\right|}{\left|\nu\right|}\right)^{l}\left|H_{m-l}\left[\sqrt{\frac{2\mu}{\nu^{*}}}\left(\nu^{*}\alpha+\mu^{*}\alpha^{*}\right)\right]\right|^{2}. \qquad (47)
\end{aligned}$$

4.2 Generalized Differential Expression Related to Two-variable Hermite Polynomials

Following we shall derive another gereralized differential expression related to two-variable Hermite Polynomials. From Eqs. (34) and (35) we can express the classical correspondence of $G_2\left(a^{\dagger},a\right)=a^{\dagger m}\hat{U}\left|0\right\rangle\left\langle 0\right|\hat{U}^{-1}a^{n}$ as

$$\begin{aligned}
g_2\left(\alpha^{*},\alpha\right) &= 2\pi Tr\left[\hat{U}\left(\mu a^{\dagger}-\sigma a\right)^{m}\left|0\right\rangle\left\langle 0\right|\left(\tau a-\nu a^{\dagger}\right)^{n}\hat{U}^{-1}\Delta\left(\alpha,\alpha^{*}\right)\right] \\
&= 2\pi\left(-\sqrt{\frac{\sigma\mu}{2}}\right)^{m}\left(-\sqrt{\frac{\nu\tau}{2}}\right)^{n}\mathrm{Tr}\left[\hat{U}H_{m}\left(\sqrt{\frac{\mu}{2\sigma}}a^{\dagger}\right)\left|0\right\rangle\left\langle 0\right|H_{n}\left(\sqrt{\frac{\tau}{2\nu}}a\right)\hat{U}^{-1}\Delta\left(\alpha,\alpha^{*}\right)\right].
\end{aligned}$$

Due to

$$H_{m}\left(x\right)=\sum_{k=0}^{[m/2]}\frac{\left(-1\right)^{k}m!}{k!\left(m-2k\right)!}\left(2x\right)^{m-2k},$$

we can obtain

$$\begin{aligned}
g_2\left(\alpha^{*},\alpha\right) &= 2\pi\left(-\frac{\mu}{2}\right)^{m}\left(-\frac{\tau}{2}\right)^{n}\sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]}\frac{\left(-\frac{\sigma}{2\mu}\right)^{k}m!}{k!\sqrt{(m-2k)!}}\frac{\left(-\frac{\nu}{2\tau}\right)^{l}n!}{l!\sqrt{(n-2l)!}} \\
&\quad\times\mathrm{Tr}\left[\hat{U}\left|m-2k\right\rangle\left\langle n-2l\right|\hat{U}^{-1}\Delta\left(\alpha,\alpha^{*}\right)\right]. \qquad (48)
\end{aligned}$$

Supposed a projection operator of the number state $\left|m\right\rangle\left\langle n\right|$ and from Eq. (37), its Weyl ordering reads as

$$\begin{aligned}
\left|m\right\rangle\left\langle n\right| &= 2\int\frac{d^{2}z}{\pi}\,\vdots\,\left\langle -z\right|\left.m\right\rangle\left\langle n\right|\left.z\right\rangle\exp\left[2\left(az^{*}-a^{\dagger}z+a^{\dagger}a\right)\right]\,\vdots \\
&= 2\int\frac{d^{2}z}{\pi}\,\vdots\,\frac{\left(-z^{*}\right)^{m}z^{n}}{\sqrt{n!m!}}\exp\left[-2\left|z\right|^{2}+2\left(az^{*}-a^{\dagger}z+a^{\dagger}a\right)\right]\,\vdots\,. \qquad (49)
\end{aligned}$$

By virtue of the following integral formula

$$\int\frac{d^{2}\beta}{\pi}\beta^{*k}\beta^{l}\exp\left[-h\left|\beta\right|^{2}+s\beta+f\beta^{*}\right]=\left(-i\right)^{k+l}h^{-\frac{k+l}{2}-1}e^{\frac{sf}{h}}H_{k,l}\left(\frac{is}{\sqrt{h}},\frac{if}{\sqrt{h}}\right)$$

we have

$$\left|m\right\rangle\left\langle n\right|=\frac{2}{\sqrt{n!m!}}\,\vdots\,H_{m,n}\left(2a^{\dagger},2a\right)\exp\left(-2a^{\dagger}a\right)\,\vdots\,. \qquad (50)$$

Therefore, considering Eqs. (32) and (50), we can see

$$\begin{aligned}
g_2\left(\alpha^{*},\alpha\right) &= 4\pi\left(-\frac{\mu}{2}\right)^{m}\left(-\frac{\tau}{2}\right)^{n}\sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]}\frac{\left(-\frac{\sigma}{2\mu}\right)^{k}m!}{k!\left(m-2k\right)!}\frac{\left(-\frac{\nu}{2\tau}\right)^{l}n!}{l!\left(n-2l\right)!} \\
&\quad\times\mathrm{Tr}\left[\,\vdots\,H_{m-2k,n-2l}\left[2\left(\sigma a+\tau a^{\dagger}\right),2\left(\mu a+\nu a^{\dagger}\right)\right]\exp\left[-2\left(\sigma a+\tau a^{\dagger}\right)\left(\mu a+\nu a^{\dagger}\right)\right]\,\vdots\,\Delta\left(\alpha,\alpha^{*}\right)\right].
\end{aligned}$$

Weyl correspondence rule in (17) tells us that the classical correspondence of $G_2\left(a^{\dagger},a\right)$ is

$$\begin{aligned}
g_2\left(\alpha^{*},\alpha\right) &= 2\left(-\frac{\mu}{2}\right)^{m}\left(-\frac{\tau}{2}\right)^{n}\exp\left[-2\left(\sigma\alpha+\tau\alpha^{*}\right)\left(\mu\alpha+\nu\alpha^{*}\right)\right] \\
&\quad\times\sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]}\frac{\left(-\frac{\sigma}{2\mu}\right)^{k}m!}{k!\left(m-2k\right)!}\frac{\left(-\frac{\nu}{2\tau}\right)^{l}n!}{l!\left(n-2l\right)!}H_{m-2k,n-2l}\left[2\left(\sigma\alpha+\tau\alpha^{*}\right),2\left(\mu\alpha+\nu\alpha^{*}\right)\right]. \qquad (51)
\end{aligned}$$

Comparing Eqs. (51) and (44), we also derive a simplified equation

$$
\begin{aligned}
& \exp\left[4\mu\tau\left|\alpha\right|^{2}+2\nu\tau\alpha^{*2}+2\sigma\mu\alpha^{2}\right]\frac{\partial^{m}}{\partial\alpha^{m}}\frac{\partial^{n}}{\partial\alpha^{*n}}\exp\left[-4\mu\tau\left|\alpha\right|^{2}-2\nu\tau\alpha^{*2}-2\sigma\mu\alpha^{2}\right] \\
= \;& 2\mu^{m}\tau^{n}\sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]}\frac{\left(-\frac{\sigma}{2\mu}\right)^{k}m!}{k!\left(m-2k\right)!}\frac{\left(-\frac{\nu}{2\tau}\right)^{l}n!}{l!\left(n-2l\right)!}H_{m-2k,n-2l}\left[2\left(\sigma\alpha+\tau\alpha^{*}\right),2\left(\mu\alpha+\nu\alpha^{*}\right)\right]. \qquad (52)
\end{aligned}
$$

The right of (52) is a summation of two-variable Hermite Polynomials, while Eq. (45) is related to the product of two single-variable Hermite Polynomials. Eqs. (45) and (52) are new generalized differential expressions related to the Hermite Polynomials. For a special case of $m=n$, Eq. (52) reduces to

$$
\begin{aligned}
& \exp\left[4\mu\tau\left|\alpha\right|^{2}+2\nu\tau\alpha^{*2}+2\sigma\mu\alpha^{2}\right]\frac{\partial^{m}}{\partial\alpha^{m}}\frac{\partial^{m}}{\partial\alpha^{*m}}\exp\left[-4\mu\tau\left|\alpha\right|^{2}-2\nu\tau\alpha^{*2}-2\sigma\mu\alpha^{2}\right] \\
= \;& 2\left(\mu\tau\right)^{m}\sum_{k,l=0}^{[m/2]}\frac{\left(-\frac{\sigma}{2\mu}\right)^{k}m!}{k!\left(m-2k\right)!}\frac{\left(-\frac{\nu}{2\tau}\right)^{l}m!}{l!\left(m-2l\right)!}H_{m-2k,m-2l}\left[2\left(\sigma\alpha+\tau\alpha^{*}\right),2\left(\mu\alpha+\nu\alpha^{*}\right)\right].
\end{aligned}
$$

For a special case of unitary operator, e.g. $\hat{U}=\exp\left[\frac{r}{2}\left(a^{\dagger2}-a^{2}\right)\right]$, we have $\tau=\mu^{*}=\cosh r$, $\sigma=\nu^{*}=\sinh r$, Eq. (52) can be simplified to be

$$
\begin{aligned}
& \exp\left[4\cosh^{2}r\left|\alpha\right|^{2}+2\sinh r\cosh r\left(\alpha^{2}+\alpha^{*2}\right)\right]\frac{\partial^{m}}{\partial\alpha^{m}}\frac{\partial^{n}}{\partial\alpha^{*n}}\exp\left[-4\cosh^{2}r\left|\alpha\right|^{2}-2\sinh r\cosh r\left(\alpha^{2}+\alpha^{*2}\right)\right] \\
= \;& 2\cosh^{m+n}r\sum_{k=0}^{[m/2]}\sum_{l=0}^{[n/2]}\frac{\left(-\frac{\tanh r}{2}\right)^{k+l}}{k!\left(m-2k\right)!}\frac{m!n!}{l!\left(n-2l\right)!}H_{m-2k,n-2l}\left[2\left(\alpha^{*}\cosh r+\alpha\sinh r\right),2\left(\alpha\cosh r+\alpha^{*}\sinh r\right)\right].
\end{aligned}
$$

5 Applications of Eq.(45) in the Study of Nonclassical Features of Quantum Light Field

For a composite system consisting of a multi-level atom (or quantum dot) coupled to a cavity and driven by a weak coherent field, quantum-optical effects can be demonstrated in the interaction processes of photon emission and absorbtion with atom between ground and excited states [20, 21, 22, 23]. In Ref. [24], author has considered a weak coherent incident field, and the quantum state of emitted light can be expressed as a series of excited coherent states $\sum_{m}C_{m}a^{\dagger m}\left|\alpha\right\rangle$ or a superposition of the different Fock states $\left|\psi\right\rangle=\sum_{n}C_{n}\left|n\right\rangle$ (due to $\left|\alpha\right\rangle=\sum_{n}\frac{\alpha^{n}}{\sqrt{n!}}\left|n\right\rangle$). Here we consider a strong coupling case, and the initial state of the incident source is a single-mode squeezed vacuum field. Only considering an effective measurement to optical field (e.g. photon-statistics methods), this multi-photon processes originated from quantum nonlinearity can be monitored by

$$
\rho\left(r,n\right)=C_{n}^{-1}a^{\dagger n}S\left(r\right)\left|0\right\rangle\left\langle 0\right|S^{-1}\left(r\right)a^{n}. \qquad (53)
$$

which can exhibit similar behavior to that of an excited quantum state (e.g. $a^{\dagger m}\left|\varphi\right\rangle$). $C_{n}=Tr\left[\rho\left(r,n\right)\right]=n!\cosh^{n}rP_{n}\left(\cosh r\right)$ is a normalized constant. $P_{n}\left(\cosh r\right)$ is the expression of the Legendre polynomials. $S\left(r\right)=\exp\left[\frac{r}{2}\left(a^{\dagger2}-a^{2}\right)\right]$ denotes a unitary operator with the following transformation identities

$$
\begin{aligned}
S^{-1}\left(r\right)aS\left(r\right) &= a\cosh r+a^{\dagger}\sinh r, \\
S^{-1}\left(r\right)a^{\dagger}S\left(r\right) &= a^{\dagger}\cosh r+a\sinh r. \qquad (54)
\end{aligned}
$$

In order to investigate the non-classical features of $\rho\left(r,n\right)$, Eq. (18) shows that its quasi-probability distribution Wigner function can be written as

$$
\begin{aligned}
W\left(\alpha,\alpha^{*}\right) &= 2\pi\mathrm{Tr}\left[\rho\left(r,n\right)\Delta\left(\alpha,\alpha^{*}\right)\right] \\
&= 2\pi C_{n}^{-1}\mathrm{Tr}\left[S\left(r\right)\left(a^{\dagger}\cosh r+a\sinh r\right)^{n}\left|0\right\rangle\left\langle 0\right|\left(a\cosh r+a^{\dagger}\sinh r\right)^{n}S^{-1}\left(r\right)\Delta\left(\alpha,\alpha^{*}\right)\right] \qquad (55)
\end{aligned}
$$

Comparing (54) with (31), we can see $\mu = \tau \to \cosh r$, $\sigma = \nu \to -\sinh r$. From Eqs. (55) and (33), thus we can obtain the Wigner function of $\rho(r, n)$,

$$\begin{aligned} W(\alpha, \alpha^*) &= \frac{1}{P_n(\cosh r)} \left(\frac{-\sinh r}{2}\right)^n \exp\left[-2\left|\alpha \sinh r - \alpha^* \cosh r\right|^2\right] \\ &\times \sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix} \frac{2^l (-\coth r)^l}{(n-l)!} \left|H_{n-l}\left[i\sqrt{\frac{2}{\tanh r}}(\alpha^* \cosh r - \alpha \sinh r)\right]\right|^2 . \end{aligned}$$

In Fig. 1, negative part in certain region of phase-space indicates an evidence of nonclassicality of the state generated by adding photon to a weak squeezed radiation field. The variance of photon-addition can exhibit different nonclassical features with the fixed squeezing value of $r = 0.2$, which dominates the quadrature distribution in the directions of $-X$ and $-Y$. The top row with three pictures shows that even photon have been added to the weak squeezed radiation field, odd photon at bottom row.

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7 Appendix: Derivation of Eq. (37)

The Glauber-Sudarshan P representation, as one of most important quantum phase space theory, is the quasiprobability distribution in which observables are expressed in normal order. By virtue of the coherent state representation $|z\rangle = \exp\left(za^\dagger - z^* a\right)|0\rangle$, the P-representation of a quantum density matrix ρ is defined as

$$\rho = \int \frac{d^2 z}{\pi} P(z, z^*) |z\rangle \langle z| . \tag{56}$$

The inverse relation of Eq. (56) is

$$P(z, z^*) = e^{|z|^2} \int \frac{d^2\beta}{\pi} \langle -\beta| \rho |\beta\rangle \exp\left(|\beta|^2 + z\beta^* - z^*\beta\right), \tag{57}$$

which was first obtained by Mehta [25]. Substituting (57) into (56) and Utilizing Weyl ordering of the coherent state projector, namely

$$|z\rangle \langle z| = 2 \vdots \exp\left[-2\left(z^* - a^\dagger\right)(z - a)\right] \vdots ,$$

we have

$$\begin{aligned} \rho &= 2 \int \frac{d^2 z}{\pi} e^{|z|^2} \int \frac{d^2\beta}{\pi} \langle -\beta| \rho |\beta\rangle \exp\left(|\beta|^2 + z\beta^* - z^*\beta\right) \vdots \exp\left[-2\left(z^* - a^\dagger\right)(z - a)\right] \vdots \\ &= 2 \int \frac{d^2\beta}{\pi} \vdots \langle -\beta| \rho |\beta\rangle \exp\left[2\left(a\beta^* - a^\dagger\beta + a^\dagger a\right)\right] \vdots . \end{aligned} \tag{58}$$

which can conveniently recast operators into their Weyl ordering.

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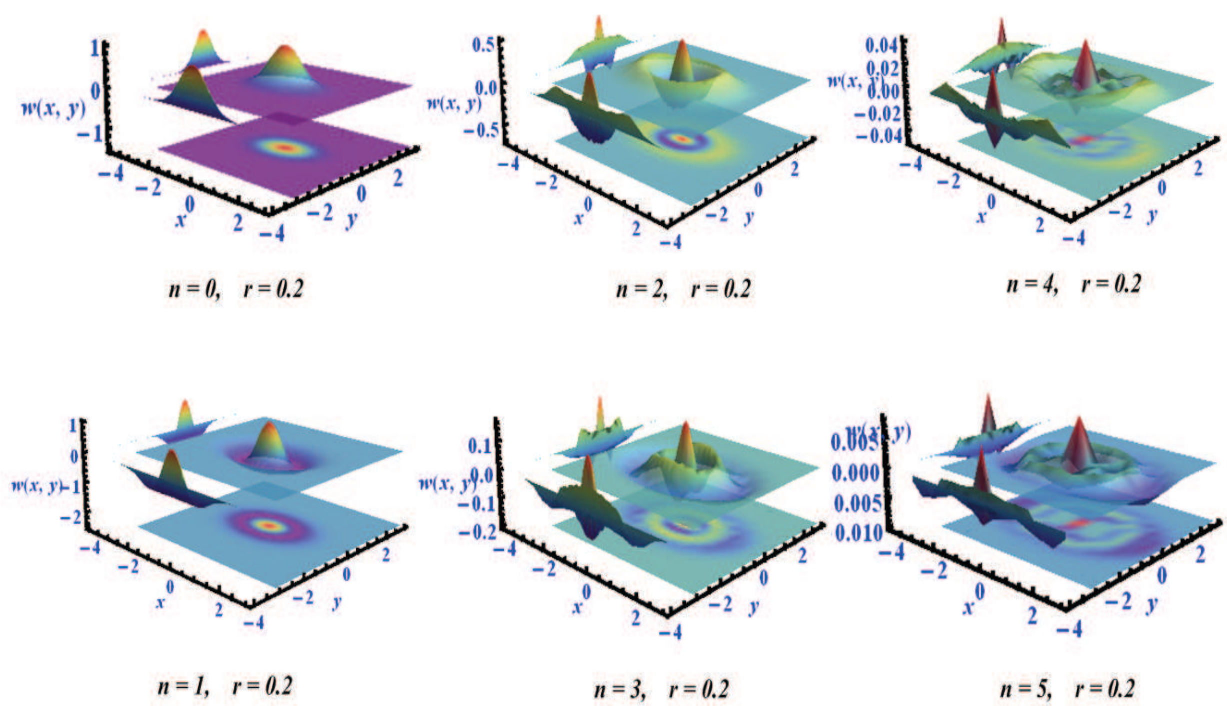

Figure 1: Wigner distributions of the excited squeezed vacuum state with fixed squeezing. In the top row, even number photons have been added to the squeezing vacuum field. The case of odd number photons has been shown in the bottom row.